EC4090: Quantitative Methods – Exercise 1
Question 9

1 Proof that OLS residuals e are distributed N(0,σ²M)

In order to show how the OLS residuals are distributed we will first try to establish a relationship between the error term in the population and the OLS residuals. To do this we start from the definition of the OLS estimates and insert this definition in the calculation of the OLS residuals.

$\boldsymbol{\beta}_{OLS} = (\mathbf{X'X})^{-1}\mathbf{X'y}$

$\mathbf{e} = \mathbf{y} - \mathbf{X}\boldsymbol{\beta}_{OLS} = \mathbf{y} - \mathbf{X}((\mathbf{X'X})^{-1}\mathbf{X'y}) = [\mathbf{I} - \mathbf{X}(\mathbf{X'X})^{-1}\mathbf{X'}]\mathbf{y} = \mathbf{My}$

$\mathbf{My} = \mathbf{M}(\mathbf{X}\boldsymbol{\beta} + \mathbf{u}) = \mathbf{MX}\boldsymbol{\beta} + \mathbf{Mu} = [\mathbf{I} - \mathbf{X}(\mathbf{X'X})^{-1}\mathbf{X'}]\mathbf{X}\boldsymbol{\beta} + \mathbf{Mu} = [\mathbf{IX} - \mathbf{IX}]\boldsymbol{\beta} + \mathbf{Mu} = \mathbf{0}\boldsymbol{\beta} + \mathbf{Mu} = \mathbf{Mu}$

$$\boxed{\mathbf{e} = \mathbf{Mu}}$$

The above expression shows that the OLS residuals are a linear transformation of the error term. According to the properties of the normal distribution described by Wooldridge (2003), a linear transformation of a normally distributed random variable is also distributed normally. Therefore we can conclude, that the OLS residuals are distributed normally. To prove that the OLS residuals are distributed N(0,σ²**M**) we have to show that the E(**e**) = 0 and that V(**e**) = σ²**M**.

1.1 Mean of e

$E(\mathbf{e}) = E(\mathbf{Mu}) = \mathbf{M}\,E(\mathbf{u}) = 0$

This follows from the assumption that the error term has a zero mean.

1.2 Variance of e

$V(\mathbf{e}) = E[(\mathbf{e}-\mathbf{0})(\mathbf{e}-\mathbf{0})'] = E(\mathbf{ee}') = E(\mathbf{Muu'M'}) = \mathbf{M}\,E(\mathbf{uu}')\,\mathbf{M} = \mathbf{M}\,V(\mathbf{u})\,\mathbf{M}' = \mathbf{M}\,\sigma^2\mathbf{I}\,\mathbf{M}' = \sigma^2\mathbf{M}$

In the above proof we use the fact that the error term u is distributed N(0,σ²**I**). Furthermore we assume that the projection matrix M is idempotent and symmetric. These two properties will be proved in the next section.

$$\boxed{\mathbf{e} \sim N(0,\sigma^2\mathbf{M})}, \text{ Q.e.D.}$$

2 Properties of the projection matrix M

In order to verify that the proof in 1.2 is correct we have to show that the projection matrix is idempotent and symmetric. But this does not only apply to the proof in 1.2. In general, a projection matrix must be idempotent because as (Davidson and MacKinnon, 2004) describe it: "*If we take any point, project it on to S(**X**), and the project it on to S(**X**) again, the second projection can have no effect at all, because the point is already in S(**X**), and so it is left unchanged*".

2.1 Proof that M is idempotent

In order to prove that M is idempotent we simply have to show that **MM** = **M**

$\mathbf{MM} = [\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'] [\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'] = \mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{I} - \mathbf{I}\,\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' + \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' =$

$= \mathbf{I} - 2\,\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' + \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' = \mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' = \mathbf{M}$

$$\boxed{\mathbf{MM} = \mathbf{M}}, \text{ Q.e.D.}$$

2.2 Proof that M is symmetric

A matrix **A** is symmetric if **A** = **A**' (cf. Chiang, 2005, p74).

$M = [\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}']' = \mathbf{I}' - [\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}']' = I - \mathbf{X}((\mathbf{X}'\mathbf{X})^{-1})'\mathbf{X}'$

The only part in above equation which differs from the definition of **M** is $((\mathbf{X}'\mathbf{X})^{-1})'$. In order to prove that M is symmetric we have to show that $((\mathbf{X}'\mathbf{X})^{-1})' = (\mathbf{X}'\mathbf{X})^{-1}$. For this purpose we use the properties of transposes and inverses described by (Chiang, 2005, p74-76).

$((\mathbf{X}'\mathbf{X})^{-1})' = ((\mathbf{X}'\mathbf{X})')^{-1} = (\mathbf{X}'\mathbf{X})^{-1}$

$$\mathbf{M} = \mathbf{M}', \text{ Q.e.D.}$$

2.3 tr(M), the trace of M

$tr(M) = tr(I) - tr(\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}') = tr(I) - tr(\mathbf{X}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1})$

X is a Txk matrix, therefore **X**' is a kxT matrix. The expression $\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ yields a TxT matrix. On the other hand $\mathbf{X}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}$ yields a kxk matrix. For this reason above equation simplifies to $tr(\mathbf{I}_T) - tr(\mathbf{I}_k) = T - k$

$$\boxed{tr(\mathbf{M}) = T - k}$$

2.4 ρ(M), the rank of M

The rank of an idempotent matrix equals its trace (cf. Wooldridge, 2003, p783).

$$\boxed{\rho(\mathbf{M}) = tr(\mathbf{M}) = T\text{-}k}$$

Bibliography

Chiang, A. (2005): Fundamental Methods of Mathematical Economics 4th ed, , McGraw-Hill (NY).

Davidson, R., MacKinnon, J. (2004): Econometric Theory and Methods, Oxford University Press

Wooldridge, J. (2003): Introductory Econometrics, Thomson South-Western